

Modified dispersion relations and (A)dS Schwarzschild Black holes

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Abstract

In this paper we investigate the impact of modified dispersion relations (MDR) on (Anti)de Sitter-Schwarzschild black holes. In this context we find the temperature of black holes can be derived with important corrections. In particular given a specific MDR the temperature has a maximal value such that it can prevent black holes from total evaporation. The entropy of the (A)dS black holes is also obtained with a logarithmic correction.

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I. INTRODUCTION

It is well known that Planck mass M_p or Planck length l_p plays an important role in quantum gravity. General belief is that the Planck length may be the minimal observable length [1, 2]. So it is natural to take the Planck length as a universal constant. But this seems leading to a puzzle, saying that the Lorentz symmetry at Planck scale is not preserved since the length is obviously not an invariant under linear Lorentz boost. One of approaches to solve this paradox is non-linear special relativity or doubly special relativity (DSR)[3], which may preserve the relativity principle and at the same time treat Planck energy as an invariant. In this framework, Lorentz symmetry is deformed such that the usual energy-momentum relation or dispersion relation may be modified at Planck scale. As pointed out in [4], a general modified dispersion relation may be written as

$$E^2 f_1^2(E; \eta) - p^2 f_2^2(E; \eta) = m_0^2, \quad (1)$$

where f_1 and f_2 are two functions of energy from which a specific formulation of boost generator can be defined. η is a dimensionless parameter (We set $\hbar = c = 1$ through the whole paper) characterizing the strength of the correction. It is also interesting to see that this formula can be incorporated into a general relativity framework, see ref. [5]. Besides this, modified dispersion relations and its implications have been greatly investigated in recent years, references can be found in [6, 7, 8, 9, 10, 11, 12, 13, 14]. In particular, it has been known that MDR may change the thermodynamical properties of black holes greatly so as to provide novel mechanism for understanding the late fate of the black hole evaporation [15, 16]. In particular, in [15] one of our authors with other collaborators studied the impact of MDR on the thermodynamics of Schwarzschild black holes. It has been found that due to the modification of dispersion relations the ordinary picture of Hawking radiation changes greatly when the mass of black holes approaches to the Planck scale. First of all, both the temperature and the entropy of black holes receive important corrections such that the temperature is bounded with a finite value rather than divergent. Secondly, such corrections may prevent the black hole from total evaporation since the heat capacity vanishes as the temperature reaches the maximal value. Such remnants of black holes may be viewed as a candidate for dark matter.

In this paper, we intend to extend above analysis to (A)dS Schwarzschild black holes. We firstly review that the usual Hawking temperature of (A)dS Schwarzschild black holes can be heuristically derived by employing the standard dispersion relation as well as extended

uncertainty principle (EUP) to radiation particles in the vicinity of horizon. Then we investigate how a general form of MDR may affect the temperature as well as the entropy of black holes in section three. The combination of both effects due to MDR and the generalized EUP (GEUP) is also presented.

II. UNCERTAINTY PRINCIPLES AND HAWKING TEMPERATURE

It has been argued in [17, 18] that there is an intrinsic uncertainty about the Schwarzschild radius R for those photons in the vicinity of horizon. With the use of this fact one can heuristically derive the the Hawking temperature of Schwarzschild black holes. However, as pointed out in Ref. [19, 20], the usual uncertainty principle can not be naively applied to the space with large length scales like as in (A)dS space. As a result, to properly obtain the temperature of (A)dS Schwarzschild black holes in this manner the usual uncertainty relations should be extended to include the effects of the cosmological constant, which may be named as the extended uncertainty principle (EUP) [21, 22]. It can be written as

$$\Delta x \Delta p \geq 1 + \beta^2 \frac{(\Delta x)^2}{L^2}, \quad (2)$$

where β is a dimensionless real constant of order one, and L is the characteristic large length scale related to the cosmological constant as $\Lambda \sim 1/L^2$. In contrast to the ordinary uncertainty relation, in EUP there exists an absolute minimum for the momentum uncertainty

$$\Delta p \geq \frac{1}{\Delta x} + \frac{\beta^2 \Delta x}{L^2} \geq \frac{2\beta}{L}. \quad (3)$$

However, it is easy to see that for a very large L , Eq.(2) goes back to the usual Heisenbergs uncertainty relation.

Now, we consider a 4-dimensional AdS black hole with the metric

$$ds^2 = -N^2 dt^2 + N^{-2} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (4)$$

where

$$N^2 = 1 + \frac{r^2}{L^2} - \frac{2GM}{r}. \quad (5)$$

The event horizon r_+ can be obtained by setting $N^2 = 0$. Now, using the standard results in statistical mechanics we expect that the energy of photons emitted from the horizon can be identified as the characteristic temperature of the Hawking radiation, namely[17, 23]

$$T \sim E = p, \quad (6)$$

where p is the momentum of photons emitted from the horizon. Next, we propose that photons emitted from the black hole satisfy the extended uncertainty principle(EUP). By modelling a black hole as a black box with linear size r_+ , the position uncertainty Δx of photons emitted from the black hole is just the horizon r_+ , i. e.

$$\Delta x \sim r_+. \quad (7)$$

while the momentum of photons in a quantum mechanical region approximately satisfy the relation $p \sim \Delta p$. Then with (3),(6)and (7), we immediately obtain the Hawking temperature

$$T_{AdS} = \frac{1}{4\pi} \left[\frac{1}{r_+} + \frac{3r_+}{L^2} \right], \quad (8)$$

where a ‘‘calibration factor’’ 4π is introduced [17] and the parameter β^2 is set to 3 for four dimensional black holes .

It is straightforward to obtain the temperature for dS black holes by setting $L^2 \rightarrow -L^2$ since the cosmological constant $\Lambda \sim 1/L^2$ [24],

$$T_{dS} = \frac{1}{4\pi} \left[\frac{1}{r_+} - \frac{3r_+}{L^2} \right]. \quad (9)$$

As pointed out in [22], it is easy to see that in the AdS case, the temperature has an absolute minimum due to EUP, while in dS case, there exists a maximal radius for black hole horizon but no minimal one. Nevertheless, in both cases the temperature will suffer from the divergency as the size of the horizon approaches to zero. This is a unsatisfactory point implying that the conventional picture of Hawking radiation may not be applicable to the late stage of black hole evaporation. To provide a more reasonable picture for this, in next section we propose to modify the usual dispersion relation for photons and discuss its possible impact on the thermodynamics of (A)dS black holes.

III. THE IMPACT OF MDR ON BLACK HOLE PHYSICS

It has been studied that the existence of a minimum length can prevent black holes from total evaporation [17, 25], where the generalized uncertainty principle(GUP) plays an essential role. In this section, we show that MDR may provide a similar mechanism to describe the late stage of (A)dS black hole radiation in a reasonable manner. As far as the uncertainty relations is concerned, we will first consider the EUP case and then turn to the GEUP one.

A. EUP case

After setting $m_0 = 0$ in (1) for photons we may rewrite the general form of modified dispersion relations as

$$E = \frac{f_2(E; \eta)}{f_1(E; \eta)} p. \quad (10)$$

As discussed in previous section, we expect that the energy of photons emitted from black holes can be identified as the Hawking temperature, but with MDR this identification will lead to a recursion relation for the black hole temperature

$$T \sim E = \frac{f_2(E; \eta)}{f_1(E; \eta)} p = \frac{f_2(T; \eta)}{f_1(T; \eta)} p. \quad (11)$$

Similarly, through the EUP in Eq.(3) the Hawking temperature of (A)dS black holes can be written as

$$T = \frac{f_2(T; \eta)}{f_1(T; \eta)} T_0, \quad (12)$$

where T_0 is given by

$$T_0 = \frac{1}{4\pi} \left[\frac{1}{r_+} \pm \frac{3r_+}{L^2} \right]. \quad (13)$$

It is easy to see that for the low energy case both functions f_1 and f_2 approach to one, the temperature in Eq.(12) goes back to the original one. However, at high energy level the temperature will receive important corrections and the expression depends on the specific form of f_1 and f_2 . For explicitness, we take the ansatz $f_1^2 = 1 - (l_p E)^2$ and $f_2^2 = 1$. Then Eq.(12) becomes

$$T^2 = \frac{1}{1 - (l_p E)^2} T_0^2 = \frac{1}{1 - (l_p T)^2} T_0^2. \quad (14)$$

From this equation we obtain the Hawking temperature as

$$T = \left[\frac{M_p^2}{2} \left(1 - \sqrt{1 - \frac{4T_0^2}{M_p^2}} \right) \right]^{1/2}, \quad (15)$$

where $M_p = l_p^{-1}$ (The other solution with plus sign ahead of the square root is ruled out as it does not provide reasonable physical meanings). From this equation it is easy to see that for large black holes where $T_0 \ll M_p$, the modified temperature goes back to the usual one $T \sim T_0$. However, when the temperature increases with the evaporation, we find that the modified temperature has a upper limit $T \leq M_p/\sqrt{2}$ and the inequality saturates when $T_0 = M_p/2$. The corresponding radius of the black hole horizon reaches a minimal value $r_+ = l_p/2\pi$. This situation is similar to the case presented in [15] while the difference here is that due to EUP, $T \sim T_0$ is also bounded from below with a minimum value for large

black holes. The existence of the minimal size of horizon implies that black hole maybe have a final stable state and this can be testified by calculating the heat capacity of the (A)dS black holes, which turns out to be

$$C_{(A)dS} = -\frac{2\pi\sqrt{1-4T_0^2/M_p^2}\left(1\pm\frac{3r_+^2}{L^2}\right)}{G\sqrt{1-T^2/M_p^2}\left(\frac{1}{r_+^2}\mp\frac{3}{L^2}\right)}, \quad (16)$$

where the upper sign and lower sign correspond to AdS and dS black holes respectively. Obviously the heat capacity vanishes as $T_0 = M_p/2$.

As a result, we argue that MDR may provide a mechanism to prevent (A)dS black holes from total evaporation such that an explosive disaster of black holes can be avoided.

In the end of this section we briefly discuss the correction to the entropy of (A)dS black holes due to MDR. It is expected that the first thermodynamical law still holds for large black holes (i.e. $8G \ll A$). Plugging Eq.(15) into $dM = TdS$, we have

$$dS = \frac{1}{2\sqrt{G}} \left[A - A\sqrt{1 - \frac{8G}{A} \left(1 + \frac{3A}{4\pi L^2}\right)^2} \right]^{-\frac{1}{2}} \left(1 + \frac{3A}{4\pi L^2}\right) dA, \quad (17)$$

where $G = 1/(8\pi M_p^2)$ and $A = 4\pi r_+^2$ is the area of the horizon. Eq.(17) can be written in a simpler way,

$$dS = \frac{1}{4\sqrt{2}G} \left[1 + \sqrt{1 - \frac{8G}{A} \left(1 + \frac{3A}{4\pi L^2}\right)^2} \right]^{1/2} dA. \quad (18)$$

Now using the condition for large black holes $8G \ll A \ll L^2$, we can obtain the entropy of the black hole,

$$S \simeq \frac{A}{4G} - \frac{1}{4} \ln \frac{A}{4G} - \frac{3A}{8\pi L^2} - \frac{9A^2}{128\pi^2 L^4}. \quad (19)$$

The first term is the conventional Bekenstein-Hawking entropy while the other terms are corrections due to MDR. It is interesting to notice that the entropy has a logarithmic term, agreement with the results obtained in string theory and loop quantum gravity [26, 27, 28, 29, 30].

B. GEUP case

In previous section we have demonstrated that an appropriate form of MDR will provide a cutoff for the temperature of (A)dS black holes. Definitely we may consider this effect in

the context of generalized EUP(GEUP), which is given by [22]

$$\Delta x \Delta p \geq 1 + \alpha^2 l_p^2 (\Delta p)^2 \pm \beta^2 \frac{(\Delta x)^2}{L^2}, \quad (20)$$

where α is a new dimensionless parameter with a “ \pm ” sign corresponding to AdS and dS black holes respectively. As shown in [22], for GEUP one has the inequality

$$\Delta p^{(-)} \leq \Delta p \leq \Delta p^{(+)}, \quad (21)$$

where

$$\Delta p^{(\pm)} = \frac{\Delta x}{2\alpha^2 l_p^2} \left[1 \pm \sqrt{1 - \frac{4\alpha^2 l_p^2}{(\Delta x)^2} \left[1 \pm \beta^2 \frac{(\Delta x)^2}{L^2} \right]} \right]. \quad (22)$$

Then, one can find that Δx has the absolute minimum

$$(\Delta x)^2 \geq \frac{4\alpha^2 l_p^2}{1 \mp 4\alpha^2 \beta^2 l_p^2 / L^2}. \quad (23)$$

Now closely following the procedure in previous section, one can obtain a modified Hawking temperature as

$$T = \frac{f_2}{f_1} T_{GEUP}, \quad (24)$$

where

$$T_{GEUP} = \frac{1}{4\pi} \frac{r_+}{2\alpha^2 l_p^2} \left[1 - \sqrt{1 - \frac{4\alpha^2 l_p^2}{r_+^2} \left[1 \pm \frac{3r_+^2}{L^2} \right]} \right], \quad (25)$$

with $\beta^2 = 3$. We notice that Eq.(24) is just an extension of Eq.(12) with $T_0 \rightarrow T_{GEUP}$. In hence, if one fixes both functions f_1 and f_2 as given in the previous section, the final modified temperature of black holes becomes

$$T_{(A)dS} = \left[\frac{M_p^2}{2} \left(1 - \sqrt{1 - \frac{4T_{GEUP}^2}{M_p^2}} \right) \right]^{1/2}. \quad (26)$$

Comparing Eqs.(25) and (26), it is interesting to notice that this specific form of MDR leads to a similar modification to the temperature as $GEUP$. Thus, both MDR and $GEUP$ may independently provide a upper limit for the temperature of black holes. In the context of $GEUP$ this maximal value is given by

$$T_{GEUP}^{max} = \frac{1}{4\pi\alpha l_p} \frac{1}{(1 \mp 12\alpha^2 l_p^2 / L^2)^{1/2}}, \quad (27)$$

where α is supposed to be non-zero and order one ($\alpha \rightarrow 0$ leading to $T_{GEUP} \rightarrow T_0$ as seen from Eq.(25)). Our above discussion indicates that the combination of MDR and $GEUP$ does not change the whole picture obtained in the subsection of EUP case greatly, but provides further modifications to the final temperature as well as the mass of black holes.

IV. SUMMARY AND DISCUSSIONS

In this paper we have investigated the impact of modified dispersion relations on the thermodynamics of (A)dS black holes. We have shown that MDR contributes corrections to the usual Hawking temperature as well as the entropy of black holes. Such corrections may play an important role in the understanding of the final fate of black holes. In particular, it provides a vanishing heat capacity at the late stage of black hole evaporation such that it can prevent black holes from total evaporation, but arriving a stable state with a minimum horizon radius and the remnant can be treated as a candidate for cold dark matter.

It is worthwhile to point out that through the paper we identify the expectation value of the energy as the temperature of black holes, which is a standard relation in statistics for radiation particles. However, as pointed out in [31] and [32], MDR may also change the statistical properties of the ensemble such that $E \sim T$ is not strictly satisfied. It may be modified as $E \sim T(1 + \delta l p^2 T^2)$. Therefore, more exact results of the Hawking temperature maybe have to take these modifications into account. However, a delicate calculation shows that this modification will not change the main picture about the final fate of black holes as we present here.

Through the paper, we choose a specific form of MDR in which both energy and momentum of particles are bounded. It is completely possible to extend our strategy to other *MDR* forms and a parallel analysis should be straightforward if such forms could provide a upper limit for the energy of particles as expected from the side of doubly special relativity. Furthermore, our discussions are applicable to (A)dS black holes in higher dimensional spacetime once the parameter β in *EUP* is properly fixed.

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